Symetrické a antisymetrické tenzory

Prostor antisymetrických tenzorů

Vnější součin

Vnitřní součin / AS-dualita

Rozštěpení antisymetrizace

Báze a komponenty AS tenzorů a forem

Totálně antisymetrické tenzory

Determinant

Vnější algebra

Prostor symetrických tenzorů

Prostor antisymetrických tenzorů

$V^{[k]} \subset V^{k}$ $\quad \dim V^{[k]} = \binom{d}{k}$ $\quad k = 0, 1, \dots, d$

$A \in V^{[k]} \equiv \forall \sigma$ permutace $[1 \dots k]$ $\quad A^{a_1 \dots a_k} = \operatorname{sign}\sigma \, A^{a_{\sigma 1} \dots a_{\sigma k}}$

antisymetrizace

$A = \mathcal{A} B \qquad A^{a_1 \dots a_k} = B^{[a_1 \dots a_k]} = \frac{1}{k!} \sum_{\sigma} \operatorname{sign}\sigma \, B^{a_{\sigma 1} \dots a_{\sigma k}}$

$A \in V^{[k]} \iff A^{a_1 \dots a_k} = A^{[a_1 \dots a_k]}$

projektor na $V^{[k]}$

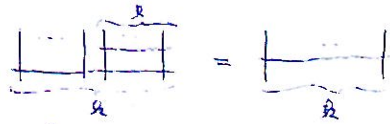
${}^{[k]}\delta \in V^{[k]}_{[k]} \qquad {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k} = \delta^{[a_1}_{b_1} \dots \delta^{a_k]}_{b_k} = \delta^{a_1}_{[b_1} \dots \delta^{a_k}_{b_k]}$

$A^{[a_1 \dots a_k]} = {}^{[k]}\delta^{a_1 \dots a_k}_{m_1 \dots m_k} A^{m_1 \dots m_k}$

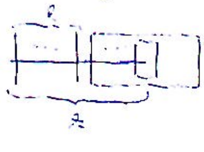
vlastnosti

${}^{[k]}\delta$ je projektor $\qquad {}^{[k]}\delta^{a_1 \dots a_k}_{m_1 \dots m_k} \, {}^{[k]}\delta^{m_1 \dots m_k}_{b_1 \dots b_k} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k} \qquad {}^{[k]}\delta^{m_1 \dots m_k}_{m_1 \dots m_k} = \dim V^{[k]}$

${}^{[k]}\delta^{a_1 \dots a_{k-l} a_{k-l+1} \dots a_k}_{b_1 \dots b_{k-l} m_1 \dots m_l} \, {}^{[l]}\delta^{m_1 \dots m_l}_{b_{k-l+1} \dots b_k} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k}$



${}^{[k]}\delta^{a_1 \dots a_l m_1 \dots m_{k-l}}_{b_1 \dots b_l m_1 \dots m_{k-l}} = \frac{(d-l)!\, l!}{(d-k)!\, k!} \, {}^{[l]}\delta^{a_1 \dots a_l}_{b_1 \dots b_l}$

 $= \frac{(d-l)!\, l!}{(d-k)!\, k!}$ 

${}^{[k]}\delta^{a_{\sigma 1} \dots a_{\sigma k}}_{b_{\sigma 1} \dots b_{\sigma k}} = {}^{[k]}\delta^{a_1 \dots a_k}_{b_1 \dots b_k}$

Vnější součin

Def:

$$\alpha^j \in V_{[p_j]} \qquad \sum p_i = p$$

$$\alpha^1 \wedge \dots \wedge \alpha^k = \frac{p!}{p_1! \dots p_k!} \mathcal{A}(\alpha^1 \dots \alpha^k)$$

$$(\alpha^1 \wedge \quad \wedge \alpha^k)_{a_1 \dots a_p} = \frac{p!}{p_1! \quad p_k!} \alpha^1_{[a_1 \dots} \alpha^2_{a_{p_1+1} \dots} \dots \alpha^k_{\dots a_p]}$$

vlastnosti:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma \qquad \text{asociativita}$$

$$\underset{p}{\alpha} \wedge \underset{q}{\beta} = (-1)^{pq} \beta \wedge \alpha \qquad \text{skew-komutat.}$$

Př: α, β 1-forma σ 2-forma ω p-forma

$$\alpha \wedge \beta = -\beta \wedge \alpha \qquad \alpha \wedge \omega = (-1)^p \omega \wedge \alpha$$

$$\sigma \wedge \omega = \omega \wedge \sigma$$

normování wedge

$$(\alpha^1 \wedge \quad \wedge \alpha^k)_{a_1 \quad a_p} =$$

$$= \sum \operatorname{sign} \sigma \; \alpha^1_{a_{\sigma^1_1} \quad a_{\sigma^1_{p_1}}} \alpha^2_{a_{\sigma^2_1} \quad a_{\sigma^2_{p_2}}} \dots \alpha^k_{a_{\sigma^k_1} \dots a_{\sigma^k_{p_k}}}$$

rozdělení čísel
(1…p) na K skupin
$\sigma^1 \dots \sigma^k$ délek $p_1 \dots p_k$

Př:

$$(\alpha \wedge \beta)_{ab} = \alpha_a \beta_b - \alpha_b \beta_a \qquad \alpha \wedge \beta = \alpha\beta - \beta\alpha$$

$$(\alpha \wedge \sigma)_{\underline{abc}} = \alpha_{\underline{a}} \sigma_{\underline{bc}} + \alpha_{\underline{b}} \sigma_{\underline{ca}} + \alpha_{\underline{c}} \sigma_{\underline{ab}}$$

$$(\underset{2}{\omega} \wedge \underset{2}{\sigma})_{\underline{abcd}} = \omega_{\underline{ab}} \sigma_{\underline{cd}} - \omega_{\underline{ac}} \sigma_{\underline{bd}} + \omega_{\underline{ad}} \sigma_{\underline{bc}}$$

$$+ \omega_{\underline{bc}} \sigma_{\underline{ad}} - \omega_{\underline{bd}} \sigma_{\underline{ac}} + \omega_{\underline{cd}} \sigma_{\underline{ab}}$$

Theorem

$$\alpha_{\mu_1\dots\mu_s}\wedge\beta_{\mu_{s+1}\dots\mu_r} =$$

$$= \sum_{1\le k_1<k_2<\dots<k_s\le r} (-1)^{k_1+\dots+k_s+\frac{s(s+1)}{2}}\, \alpha_{\mu_{k_1}\dots\mu_{k_s}}\, \beta_{\underbrace{\mu_1\dots\mu_r}_{\text{mimo } \mu_{k_1}\dots\mu_{k_s}}}$$

Důkaz:

$$\sum_{1\le k_1<\dots<k_s\le r} (-1)^{k_1+\dots+k_s+\frac{s(s+1)}{2}}\, \alpha_{\mu_{k_1}\dots\mu_{k_s}}\, \beta_{\underbrace{\mu_1\dots\mu_r}_{\text{mimo } \mu_{k_1}\dots\mu_{k_s}}} =$$

$$= \sum_{1\le k_1<\dots<k_s\le r} (-1)^{\sigma}\, \alpha_{\mu_{\sigma_1}\dots\mu_{\sigma_s}}\, \beta_{\mu_{\sigma_{s+1}}\dots\mu_{\sigma_r}} =$$

$$\sigma : \begin{bmatrix} 1 & 2 & \dots & s & s+1 & \dots & r \\ k_1 & k_2 & \dots & k_s & \underbrace{1 & \dots & r}_{\text{mimo } k_1\dots k_s} \end{bmatrix}$$

$$= \frac{1}{s!}\,\frac{1}{(r-s)!} \sum_{\sigma\in P_{1\dots r}} (-1)^{\sigma}\, \alpha_{\mu_{\sigma_1}\dots\mu_{\sigma_s}}\, \beta_{\mu_{\sigma_{s+1}}\dots\mu_{\sigma_r}}$$

$$= \binom{r}{s}\, \alpha_{[\mu_1\dots\mu_s}\, \beta_{\mu_{s+1}\dots\mu_r]} =$$

$$= \alpha_{\mu_1\dots\mu_s}\wedge\beta_{\mu_{s+1}\dots\mu_r}$$

Theorem

$$\alpha_{\mu_0}\wedge\beta_{\mu_1\dots\mu_r} = \sum_{0\le k\le r} (-1)^{k}\, \alpha_{\mu_k}\, \beta_{\underbrace{\mu_0\dots\mu_r}_{\text{mimo } \mu_k}}$$

$$\alpha_{\mu_1\mu_2}\wedge\beta_{\mu_3\dots\mu_r} = \sum_{1\le k<l\le r} (-1)^{k+l+1}\, \alpha_{\mu_k\mu_l}\, \beta_{\underbrace{\mu_1\dots\mu_r}_{\text{mimo } k,l}}$$

Vnitřní součin / AS-dualita

Def:

$$V_{[k]} \times V^{[k]} \to \mathbb{R}$$

$$\omega \bullet W = \frac{1}{k!} C^{\text{vše}}\, \omega W = \frac{1}{k!}\, \omega_{\underline{a_1 \ldots a_k}} W^{\underline{a_1 \ldots a_k}} =$$

zobecnění na

$V_{[k]} \times V^{[l]}$ – suma přes všechny indexy co jsou k dispozici

Zúžení s vektorem

$$u \cdot \omega \equiv u \lrcorner \omega = \iota_u \omega = u \bullet \omega$$

$$(u \cdot \omega)_{a_1 \ldots} = u^a \omega_{a a_1 \ldots}$$

Zúžení je (alg.) derivace vůči $\wedge$

$$u \cdot (\underset{p}{\alpha} \wedge \underset{q}{\beta}) = (u \cdot \alpha) \wedge \beta + (-1)^p \alpha \wedge (u \cdot \beta)$$

$$u \cdot (\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \ldots) =$$

$$(u \cdot \alpha^1) \wedge \alpha^2 \wedge \alpha^3 \wedge \ldots + (-1)^{p_1} \alpha^1 \wedge (u \cdot \alpha^2) \wedge \alpha^3 \wedge \ldots$$

$$+ (-1)^{p_1 + p_2} \alpha^1 \wedge \alpha^2 \wedge (u \cdot \alpha^3) \wedge \ldots + \ldots$$

jednoznačnost operace

operace ι_a na formách splňující

$\iota_a(\omega + \sigma) = \iota_a \omega + \iota_a \sigma$

$\iota_a(\omega \wedge \sigma) = (\iota_a \omega) \wedge \sigma + (-1)^p \omega \wedge \iota_a \sigma$

$\iota_a f = 0$ $\quad$ f 0-forma

$\iota_a \alpha = a \cdot \alpha$ $\quad$ α 1-forma

je už jednoznačně určená

$\iota_a \omega = a \cdot \omega$

$\Leftarrow$ rozklad obecné ω na elementární členy

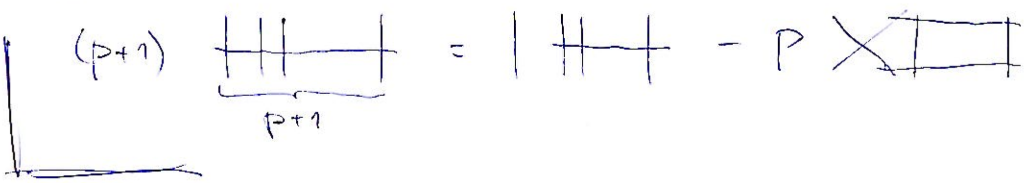
Rozštěpení antisymetrizace

$$\alpha_a \wedge \omega_{a_1 \dots a_p} = (p+1)\, \alpha_{[a}\, \omega_{a_1 \dots a_p]}$$

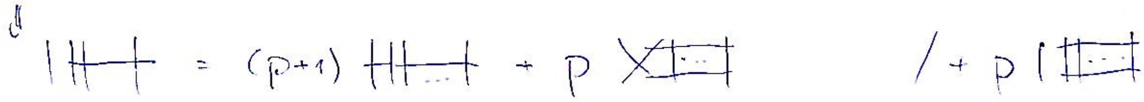
$$= \alpha_a \omega_{a_1 \dots a_p} - \alpha_{a_1} \omega_{a a_2 \dots a_p} - \alpha_{a_2} \omega_{a_1 a \dots a_p} \; \vdots \; - \alpha_{a_p} \omega_{a_1 a_2 \dots a} \quad \rightarrow \quad - \alpha_{a_1} \omega_{a a_2 \dots a_p} + \alpha_{a_2} \omega_{a a_1 \dots a_p} \; \vdots \; (-1)^p \alpha_{a_p} \omega_{a a_1 \dots a_{p-1}}$$

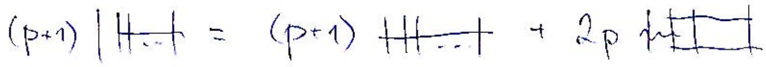
$$= \alpha_a \omega_{a_1 \dots a_p} - p\, \alpha_{[a_1}\, \omega_{|a| a_2 \dots a_p]}$$

$\Downarrow$



$\Downarrow$



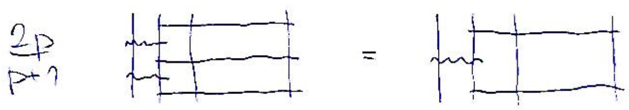


$\Downarrow$

$$\delta_{(1)[p]} = A + S$$

platí:

$$\delta_{(1)[p]}^2 = \delta_{(1)[p]} \qquad A^2 = A \qquad A \circ S = 0 \qquad \Rightarrow \qquad S^2 = S$$



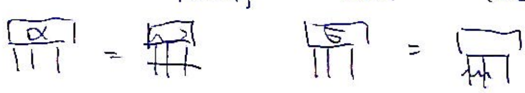
$$S^2 = S$$

Př: $p=2$

$\omega \in V_{(1)[2]}$ $\qquad \omega_{amn} = \omega_{a[mn]}$

$$\omega_{amn} = \alpha_{amn} + \frac{4}{3} \sigma_{a[mn]}$$

kde $\alpha_{amn} = \omega_{[amn]}$ $\qquad \sigma_{abm} = \omega_{(ab)m}$



Báze a komponenty AS tenzorů a forem

báze v prostorech $V^{[p]}$ a $V_{[p]}$

$V \quad e_{k}^{a} \qquad V^{*} \quad e^{k}_{a} \qquad$ duální $\quad e^{a}_{k}\, e_{a}^{l} = \delta_{k}^{l}$

$V^{[p]} \quad e_{k_1}^{[a_1} \cdots e_{k_p}^{a_p]} \qquad e_{[k_1} \cdots e_{k_p]} \qquad \mathcal{A}(e_{k_1} \ldots e_{k_p})$ tenzorová

$e_{k_1 \ldots k_p} = e_{k_1} \wedge \ldots \wedge e_{k_p} = p!\,\mathcal{A}(e_{k_1} \ldots e_{k_p})$ AS-báze

nezávislost zajištěna podm. $k_1 < \ldots < k_p$

$V_{[p]} \quad e^{[k_1}_{[a_1} \quad e^{k_p}_{a_p]} \qquad e^{[k_1} \ldots e^{k_p]} \qquad \mathcal{A}(e^{k_1} \ldots e^{k_p})$

$e^{k_1 \ldots k_p} = e^{k_1} \wedge \ldots \wedge e^{k_p} = p!\,\mathcal{A}(e^{k_1} \ldots e^{k_p})$ formové

opět $k_1 < \; < k_p$

ortonormalita

$$e^{k_1 \ldots k_p} \cdot e_{l_1 \ldots l_p} = \delta^{k_1}_{l_1} \cdots \delta^{k_p}_{l_p} \qquad k_i, l_i \text{ uspořádané}$$

souřadnice AS-tenzorů a forem

$\omega \in V_{[p]} \qquad \omega = \omega_{a_1 \; a_p}\, e^{a_1} \; e^{a_p}$ $\quad$ tenz. komponenty

$= \omega_{a_1 \; a_p}\, e^{[a_1} \cdots e^{a_p]}$ $\quad$ AS

$= \sum\limits_{a_1 < \ldots < a_p} \omega_{a_1 \ldots a_p}\, p!\,\mathcal{A}(e^{a_1} \ldots e^{a_p})$ $\quad$ uspořádání + permut.

$= \sum\limits_{a_1 < \; < a_p} \omega_{a_1 \ldots a_p}\, e^{a_1 \ldots a_p}$

$W \in V^{[p]} \qquad W = W^{a_1 \; a_p}\, e_{a_1} \; e_{a_p}$

$= \sum\limits_{a_1 < \; < a_p} W^{a_1 \; a_p}\, e_{a_1 \ldots a_p}$

$$\omega \cdot W = \sum_{\substack{a_1 < \ldots < a_p \\ b_1 < \; < b_p}} \omega_{a_1 \; a_p}\, \underbrace{e^{a_1 \ldots a_p} \cdot e_{b_1 \; b_p}}_{\delta^{a_1 \ldots}_{b_1 \ldots}}\, W^{b_1 \ldots b_p}$$

$$= \sum_{a_1 < \; < a_p} \omega_{a_1 \ldots a_p} W^{a_1 \ldots a_p} = \frac{1}{p!}\, \omega_{a_1 \ldots a_p} W^{a_1 \ldots a_p}$$

souřadnicové báze

$e^{i} = dx^{i} \qquad e_i = \dfrac{\partial}{\partial x^{i}}$

$e^{k_1 \ldots k_p} = dx^{k_1} \wedge \ldots \wedge dx^{k_p} \qquad e_{k_1 \ldots k_p} = \dfrac{\partial}{\partial x^{k_1}} \wedge \ldots \wedge \dfrac{\partial}{\partial x^{k_p}}$

Totálně antisymetrické tenzory

$\dim V^{[d]} = 1 \qquad \dim V_{[d]} = 1$

isomorfnost s $\mathbb{R}$ - skrze volbu báze

báze

$V_{[d]} \qquad \epsilon = e^{1\dots d} = e^1 \wedge \dots \wedge e^d = d!\,\mathcal{A}(e^1 \dots e^d)$

$V^{[d]} \qquad e = e_{1\dots d} = e_1 \wedge \dots \wedge e_d = d!\,\mathcal{A}(e_1 \dots e_d)$

$\epsilon \bullet e = 1 \qquad \epsilon_{a_1 \dots a_d}\, e^{a_1 \dots a_d} = d!$

souřadnice x^k

$e = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^d} \qquad \epsilon = dx^1 \wedge \dots \wedge dx^d$

souřadnice v $V^{[d]}$ a $V_{[d]}$

$\omega = \omega_{1\dots d}\, \epsilon \qquad W = W^{1\dots d}\, e$

$\omega \bullet W = \omega_{1\dots d}\, W^{1\dots d}$

transformace souřadnic

$\omega = \omega_{1\dots d}\, \epsilon = \omega'_{1\dots d}\, \epsilon' \qquad e^k = T^k_{l'}\, e^{l'}$

$\omega'_{1\dots d} = T^{k_1}_{1'} \dots T^{k_d}_{d'}\, \omega_{k_1 \dots k_d} \qquad dx^k = \frac{\partial x^k}{\partial x'^{l'}}\, dx'^{l'} \qquad T^k_{l'} = \frac{\partial x^k}{\partial x'^{l'}}$

$= \sum_{\substack{\sigma \text{ permut} \\ 1\dots d}} T^{\sigma_1}_{1'} \dots T^{\sigma_d}_{d'}\, \omega_{\sigma_1 \dots \sigma_d} = \sum_{\substack{\sigma \text{ perm} \\ 1\dots d}} \operatorname{sign}\sigma\, T^{\sigma_1}_{1'} \dots T^{\sigma_d}_{d'}\, \omega_{1\dots d}$

$= (\det T^k_{k'})\, \omega_{1\dots d}$

$W'^{1\dots d} = (\det T^k_{l'})^{-1}\, W^{1\dots d}$

inverze

$^{-1}: V_{[d]} \leftrightarrow V^{[d]} \qquad {}^{-1} \circ {}^{-1} = \mathrm{id}$

$\omega^{-1} \bullet \omega = 1 \qquad W^{-1} \bullet W = 1$

$\omega^{-1\, 1\dots d} = (\omega_{1\dots d})^{-1} \qquad W^{-1}{}_{1\dots d} = (W^{1\dots d})^{-1}$

nelineární!

$(\Omega\omega)^{-1} = \frac{1}{\Omega}\,\omega^{-1} \qquad (\Omega W)^{-1} = \frac{1}{\Omega}\, W^{-1}$

Determinant

determinant operátoru (tenzoru z V^1_1)

$$\det A = A^{a_1}_{[a_1} \cdots A^{a_d}_{a_d]} = \delta^{[b_1 \dots b_d]}_{a_1 \dots a_d} A^{a_1}_{b_1} \cdots A^{a_d}_{b_d}$$

souřadnice

$\det A = A^{a_1}_{[a_1} \cdots A^{a_d}_{a_d]}$ přispívají pouze různé a_i tj. permutace

$= \frac{1}{d!} \sum_{\sigma \text{ perm } 1 \dots d} \operatorname{sign}\sigma \, A^{a_1}_{a_{\sigma_1}} \cdots A^{a_d}_{a_{\sigma_d}}$ rozepsání AS, dále $a_i \to \alpha_i$

$= \frac{1}{d!} \sum_{\alpha, \sigma \text{ perm } 1 \dots d} \operatorname{sign}\sigma \, A^{\alpha_1}_{\alpha_{\sigma_1}} \cdots A^{\alpha_d}_{\alpha_{\sigma_d}}$ přeuspořádat permut. α^{-1}

$= \frac{1}{d!} \sum_{\alpha, \sigma \text{ perm } 1 \dots d} \operatorname{sign}\sigma \, A^1_{(\alpha\circ\sigma\circ\alpha^{-1})_1} \cdots A^d_{(\alpha\circ\sigma\circ\alpha^{-1})_d}$ přeznačení $\varrho = \alpha\circ\sigma\circ\alpha^{-1}$

$= \frac{1}{d!} \sum_{\alpha \text{ perm}} \sum_{\varrho \text{ perm}} \operatorname{sign}\varrho \, A^1_{\varrho_1} \cdots A^d_{\varrho_d} =$

$= \det A^k_l$

$$\det A'^{k'}_{l'} = \det(T^{-1\,k'}_{m} A^m_n T^n_{l'}) = \det T^{-1\,k}_{l'} \det A^m_n \det T^k_l = \det A^k_l$$

Vnější algebra

lze definovat přímý součet prostorů $V_{[p]}$

$$\Lambda = \bigoplus_{p=0}^{d} V_{[p]}$$

nehomogenní formy

$$\omega = \underset{V_{[0]}}{{}^{(0)}\omega} + \underset{V_{[1]}}{{}^{(1)}\omega} + \dots + \underset{V_{[d]}}{{}^{(d)}\omega}$$

nelze rozumně používat indexové notace

gradovaná algebra

prvky $V_{[p]}$ homogenní

p sudé → sudé formy

p liché → liché formy

netriviální komutace

pro homogenní prvky

$$\omega\wedge\sigma = (-1)^{pq}\,\sigma\wedge\omega$$

pro nehomogenní prvky - linearitou

Př: $\omega = \underset{0}{c} + \underset{1}{\alpha} + \underset{3}{\kappa}$

$\sigma = \underset{1}{\beta} + \underset{2}{\lambda}$

$$\omega\wedge\sigma = (c+\alpha+\kappa)\wedge(\beta+\lambda) = \beta\wedge(c-\alpha-\kappa) + \lambda\wedge(c+\alpha+\kappa)$$

$$= (\beta+\lambda)c + (-\beta+\lambda)\wedge\alpha + (-\beta+\lambda)\wedge\kappa$$

Prostor symetrických tenzorů

$V^{(k)} \subset V^{k}$ $\qquad \dim V^{(k)} = \binom{k+d-1}{k}$ $\qquad k \in \mathbb{N}_0$

$A \in V^{(k)} \equiv \forall \sigma$ permutace$\{1, \dots, k\}$ $\quad A^{a_1 \dots a_k} = A^{a_{\sigma_1} \dots a_{\sigma_k}}$

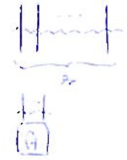
symetrizace

$A = \mathcal{S} B \qquad A^{a_1 \dots a_k} = B^{(a_1 \dots a_k)} = \frac{1}{k!} \sum_{\sigma} B^{a_{\sigma_1} \dots a_{\sigma_k}}$

$A \in V^{(k)} \Leftrightarrow A^{a_1 \dots a_k} = A^{(a_1 \dots a_k)}$

projektor na $V^{(k)}$

$\overset{(k)}{\delta} \in V^{(k)}_{(k)} \qquad \overset{(k)}{\delta}{}^{a_1 \dots a_k}_{b_1 \dots b_k} = \delta^{(a_1}_{b_1} \cdots \delta^{a_k)}_{b_k} = \delta^{a_1}_{(b_1} \cdots \delta^{a_k}_{b_k)}$



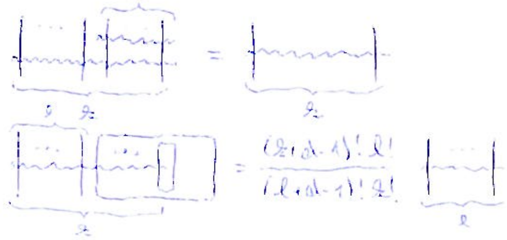
$A^{(a_1 \dots a_k)} = \overset{(k)}{\delta}{}^{a_1 \dots a_k}_{m_1 \dots m_k} A^{m_1 \dots m_k}$

vlastnosti

$\overset{(k)}{\delta}$ je projektor: $\quad \overset{(k)}{\delta}{}^{a_1 \dots a_k}_{m_1 \dots m_k} \overset{(k)}{\delta}{}^{m_1 \dots m_k}_{b_1 \dots b_k} = \overset{(k)}{\delta}{}^{a_1 \dots a_k}_{b_1 \dots b_k} \qquad \overset{(k)}{\delta}{}^{m_1 \dots m_k}_{m_1 \dots m_k} = \dim V^{(k)}$

$\overset{(k)}{\delta}{}^{a_1 \dots a_{k-l}\, a_{k-l+1} \dots a_k}_{b_1 \dots b_{k-l}\, m_1 \dots m_l} \overset{(l)}{\delta}{}^{m_1 \dots m_l}_{b_{k-l+1} \dots b_k} = \overset{(k)}{\delta}{}^{a_1 \dots a_k}_{b_1 \dots b_k}$

$\overset{(k)}{\delta}{}^{a_1 \dots a_l\, m_1 \dots m_{k-l}}_{b_1 \dots b_l\, m_1 \dots m_{k-l}} = \frac{(k+d-1)!\, l!}{(l+d-1)!\, k!} \overset{(l)}{\delta}{}^{a_1 \dots a_l}_{b_1 \dots b_l}$



$\overset{(k)}{\delta}{}^{a_{\sigma_1} \dots a_{\sigma_k}}_{b_{\sigma_1} \dots b_{\sigma_k}} = \delta^{a_1 \dots a_k}_{b_1 \dots b_k}$